Student Number

2023 GLENWOOD HIGH SCHOOL

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions

* Reading Time – 10 minutes
* Working time – 2 hours
* Write using a black pen
* NESA approved calculators may be used
* A reference sheet is provided
* For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 5)

* Attempt Questions 1-10
* Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 14)

* Attempt Questions 11 – 14
* Allow about 1 hour and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1 – 10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. Find the Cartesian equation of the curve represented by the parametric equations:

$$x = 4 - t$$
$$y = 3t^3$$

A. $y = 64 - 48x + 12x^2 - x^3$

B. $y = 64 + 48x + 12x^2 + x^3$

C. $y = 192 + 144x + 36x^2 + 3x^3$

D. $y = 192 - 144x + 36x^2 - 3x^3$

2. If α, β and γ are the roots of the equation $3x^3 + 2x^2 - x + 5 = 0$, evaluate $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

A. $\frac{5}{2}$

B. $\frac{1}{5}$

C. $\frac{2}{5}$

D. $\frac{5}{3}$

3. What is the domain and range of $y = 4\cos^{-1}\frac{3x}{2}$?

A. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$, Range: $-4\pi \leq y \leq 4\pi$.

B. Domain: $-\frac{3}{2} \leq x \leq \frac{3}{2}$, Range: $-4\pi \leq y \leq 4\pi$.

C. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$, Range: : $0 \leq y \leq 4\pi$.

D. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$, Range: $0 \leq y \leq 4$.

4.

$$\frac{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}{\cot\frac{\theta}{2}+\tan\frac{\theta}{2}} =$$

A. $Cos\,\theta$

B. $Sec\,\theta$

C. $Tan\,\theta$

D. $Cot\,\theta$

5. A spherical metal object is heated so that its radius is expanding at the rate of 0.03 mm per second. At what rate will its volume be increasing when the radius is 3.2 mm?

A. $0.4563\pi\ \mathrm{mm}^3/\mathrm{s}$

B. $1.2288\pi\ \mathrm{mm}^3/\mathrm{s}$

C. $1.9976\pi\ \mathrm{mm}^3/\mathrm{s}$

D. $2.8563\pi\ \mathrm{mm}^3/\mathrm{s}$

6. Ken, Angela and five other people go into a shop one at a time. The number of ways the seven people can go into the shop if Ken goes into the shop after Angela is

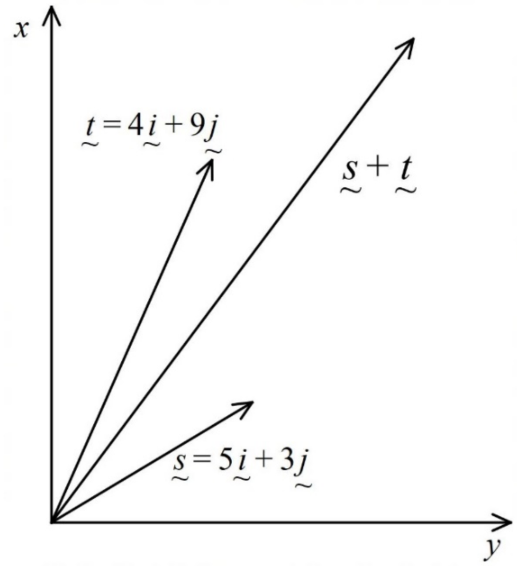
A. 21

B. 120

C. 2520

D. 5040

7. The diagram below shows the vectors $\underset{\sim}{s}$ and $\underset{\sim}{t}$ along with the vector $\underset{\sim}{s} + \underset{\sim}{t}$.



What is the value of $|\underset{\sim}{s} + \underset{\sim}{t}|$?

A. 12

B. 15

C. 16

D. 20

8. It is given that,

$$\int \frac{a}{b + x^2} dx = 2 \tan^{-1} \frac{x}{\sqrt{2}} + c$$

where $a, b \in \mathbb{R}$.

What is the value of a?

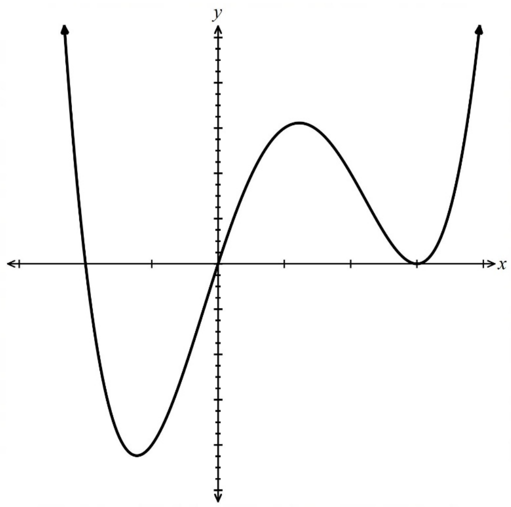
A. $\sqrt{2}$

B. 2

C. $2\sqrt[4]{2}$

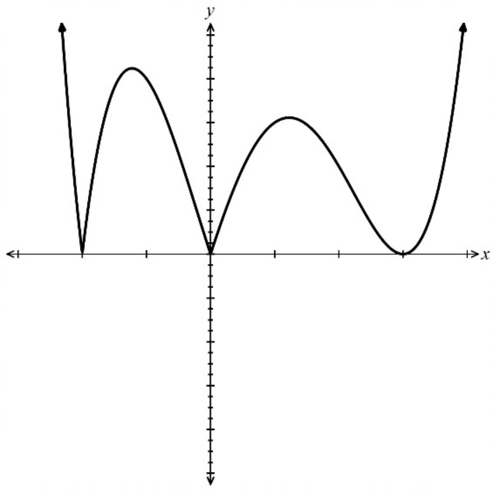
D. $2\sqrt{2}$

9. The graph of $y = f(x)$ is shown below.

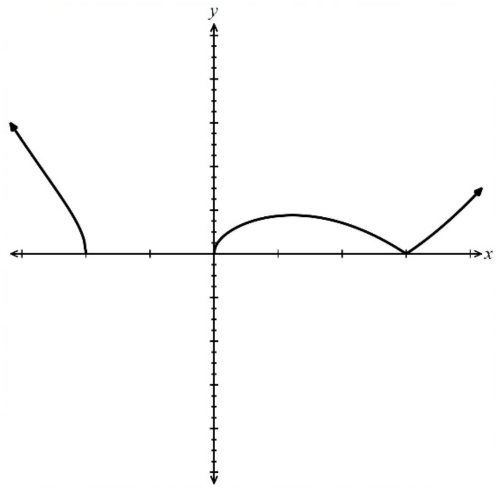


Which graph shows $y^2 = f(x)$?

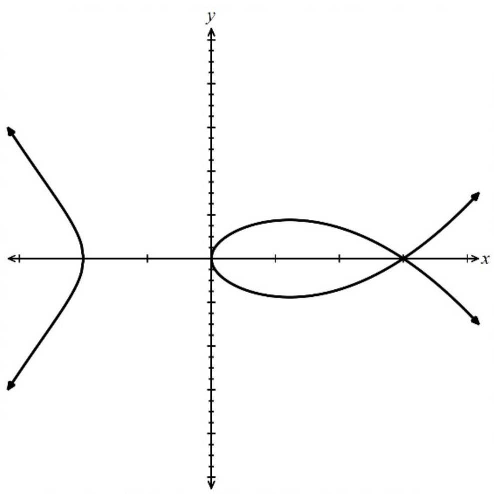
A.



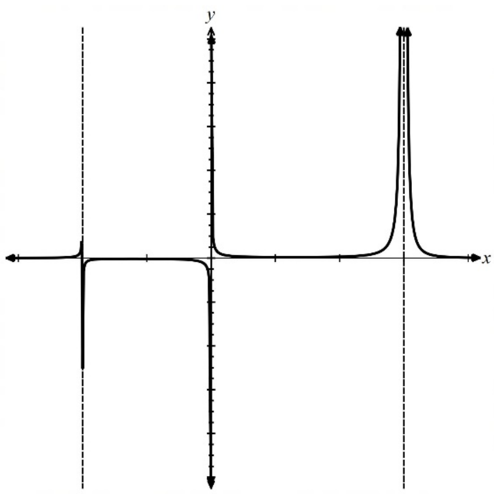
B.



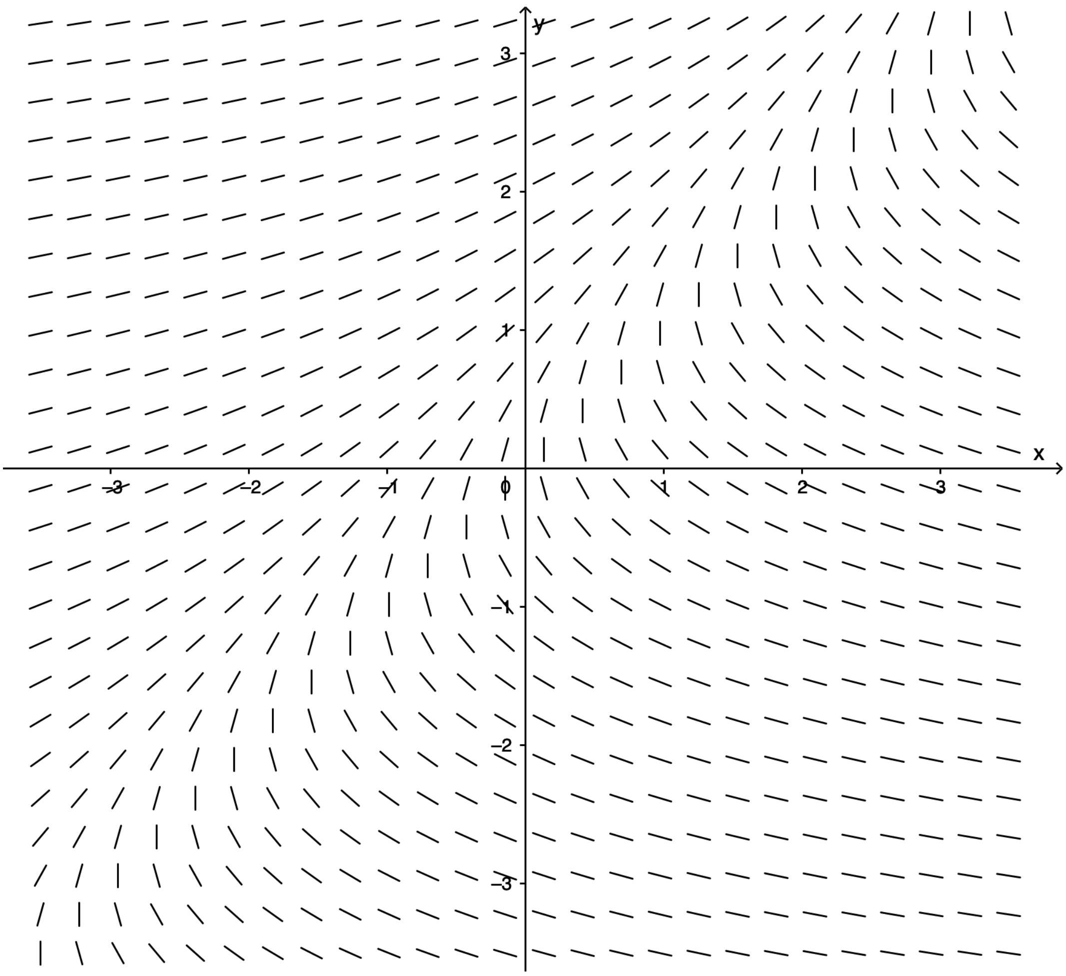
C.



D.



10. A differential equation has a direction field shown below.



A solution curve to this differential equation includes $(1, -3)$.

Which one of the following points will the solution curve also include?

A. $(0, -1)$

B. $(-1, 1)$

C. $(0, 0.75)$

D. $(1, 2)$

Section II

60 marks

Attempt Questions 11 – 14.

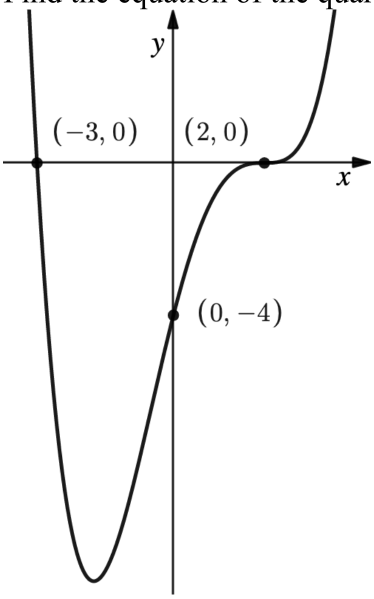
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet.

(a) Find the equation of the quartic shown below. **2**



(b) i) Write down the expansion of $(1-x)^5$. **1**

ii) Hence, find the term with x^2 in $(2x-3)^2(1-x)^5$. **2**

(c) Use sums to products to simplify and hence solve the equation. **3**

$$\sin 5x + \sin x = \sin 3x \ , \ 0 \le x \le \pi.$$

(d) Evaluate the definite integral using the substitution $u = log_e x$. **3**

$$\int_e^{e^2} \frac{2}{x(log_e x)^2}\, dx$$

Question 11 continues on page 9

(e) Records show that 64% of students at a school travelled to and from school by bus. A group of 100 students at the school are taken to determine the number of students who travel to and from school by bus.

i) Evaluate $E(X)$ and $\sigma(X)$ **2**

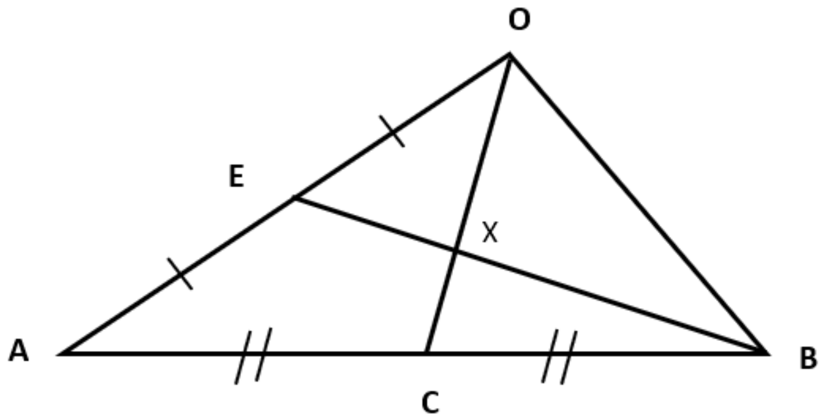
ii) Use the table below of $P(Z<z)$, where Z has a standard normal distribution, to estimate the probability that a sample of 100 students will contain at least 58 and at most 64 students who travel to and from school by bus. **2**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

End of Question 11

Question 12 (15 marks) Use a new writing booklet.

(a) The diagram below shows triangle AOB, where $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$. OC and EB intersect at X as shown. E bisects $\overrightarrow{OA}$ while C bisects $\overrightarrow{AB}$.



i) Find $\overrightarrow{OC}$ in terms of $\underline{a}$ and $\underline{b}$. **1**

ii) If $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, prove that triangle AOB is isosceles. **2**

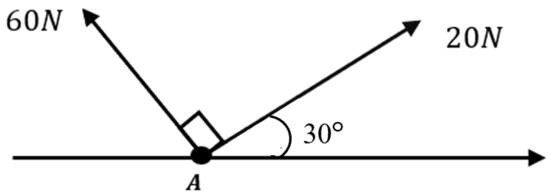
(b) Express $9\sin x + 40\cos x$ in the form $A\sin(x+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$, and hence or otherwise solve $9\sin x + 40\cos x = 6$ for $0 \le x \le \pi$. **4**

(Leave your answer for x and α in radians, correct to three significant figures).

(c) Find the equation of the function which passes through the point $\left(1, \frac{\sqrt{6}}{3}\right)$, which satisfies the differential equation $\frac{dy}{dx} = \frac{x^3+1}{xy}$. **3**

Question 12 continues on page 11

(d) Forces of $60N$ and $20N$ act on an object, considered point A, as shown in the diagram.



i) Find the net force in the horizontal direction. **1**

ii) The vector sum of these forces acting on the object at point A is called the resultant force. **2**

Find the resultant force vector in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$

(e) Find the derivative of the inverse function $f^{-1}(x)$ of $y = f(x) = 4x(x+5)^6$ in terms of y. **2**

End of Question 12

Question 13 (15 marks) Use a new writing booklet.

(a) Given that $f(x) = \tan^{-1}(e^{2x-1})$, find the value of $f'\left(\frac{1}{2}\right)$. **2**

(b) i) Consider the identity $\cos 3\theta = 4cos^3\theta - 3\cos\theta$. By letting $x = \cos\theta$, in the cubic equation $8x^3 - 6x - 1 = 0$, show that $\cos 3\theta = \frac{1}{2}$. **1**

ii) Hence prove that $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$ **2**

(c) i) Sketch the graph of $y = \sin^{-1}(2x-1)$ **2**

ii) Solve $\sin^{-1}(2x-1) = \cos^{-1}x$ **3**

(d) Jack sets up a new fish farm with 2500 fish. The differential equation for P, the population of the fish, is given by $\frac{dP}{dt} = 0.00016P(10000 - P)$.

i) Show that $\frac{1}{0.00016P(10000-p)} = \frac{5}{8}\left(\frac{1}{P} + \frac{1}{10000-P}\right)$ **1**

ii) By solving the differential equation $\frac{dP}{dt}$, show that the fish population, P in terms of t is given by the equation **4**

$$P = \frac{10000}{1+3e^{-1.6t}}$$, where t is measured in years.

Hence find the population of fish after 3 years.

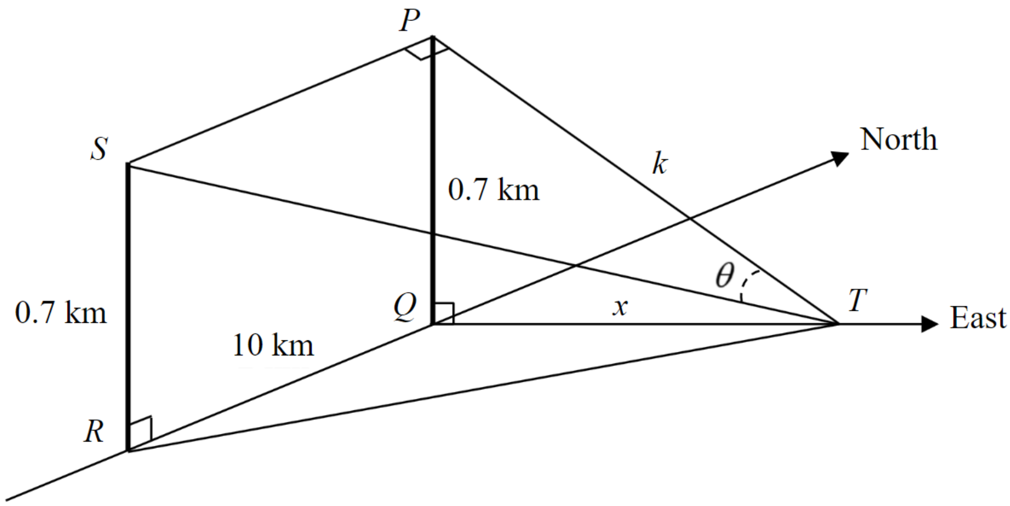
End of Question 13

Question 14 (15 marks) Use a new writing booklet.

(a) Use the principle of mathematical induction to show that for all integers $n \geq 1$, **3**

$1 + 5 + 25 + \ldots\ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

(b) PQ and SR are two towers of height 0.7 km. Q is 10 km due North of R. A vehicle at T is travelling due East away from Q at a constant speed of 10 km/h. Let the distance QT be x, the distance PT be k and $\angle PTS = \theta$.



i) Show that: **2**

$$\frac{dk}{dt} = \frac{10x}{\sqrt{x^2 + 0.49}}$$

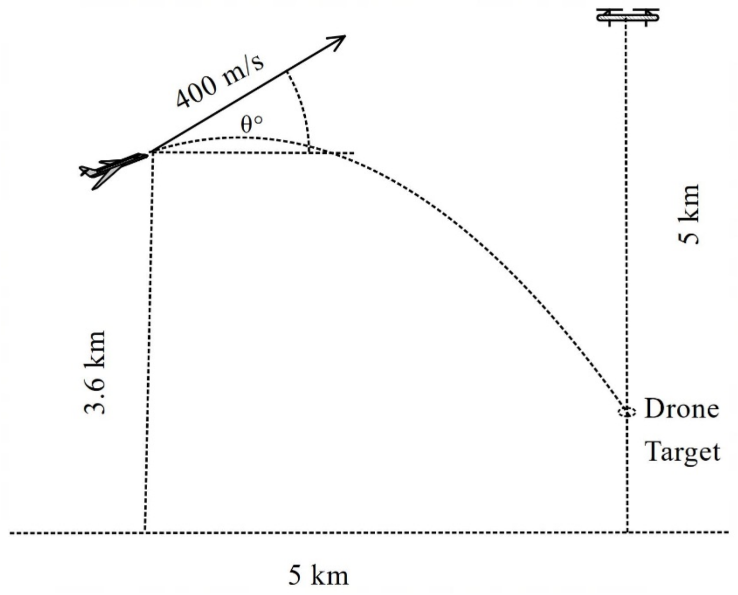
ii) By first finding an expression for k in terms of θ, show that

$$\frac{dk}{d\theta} = -10\ \text{cosec}^2\theta$$ **2**

iii) Find the exact rate at which θ is changing when the vehicle is 2.4 km from Q. **3**

Question 14 continues on page 14

(c) A drone, which is hovering at a height of 5 km, releases a target object which falls under gravity. At the same time, a jet, which is at a height of 3.6 km and is 5km west of the drone, fires a projectile at a speed of 400 m/s at an angle of θ° to the horizontal toward the target.



Using a point on the ground directly below the jet as the origin, the positions of the projectile and target at time *t* seconds after the projectile is launched are as follows:

Projectile

$$\underset{\sim}{p}(t) = \begin{pmatrix} 400t\cos\theta \\ 3600 + 400t\sin\theta - 5t^2 \end{pmatrix}$$

Drone Target

$$\underset{\sim}{d}(t) = \begin{pmatrix} 5000 \\ 5000 - 5t^2 \end{pmatrix}$$

i) Calculate the size of angle θ, if the projectile is to hit the target. **2**

ii) Determine how many seconds after the projectile is fired that it hits the target, the height of the target at that time, and the speed at which the target is falling when it is struck by the projectile. **3**

End of Examination

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1) $x = 4-t$ —① D
$y = 3t^3$ —②
from ① $t = 4-x$
$y = 3(4-x)^3$
$= 3(64 - 48x + 12x^2 - x^3)$
$= 192 - 144x + 36x^2 - 3x^3$

2). $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ B

$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{c/a}{-d/a}$

$= \frac{-1/3}{-5/3}$

$= 1/5$

3). $x \in [-2/3, 2/3]$ C
$y \in [0, 4\pi]$

4) $\frac{1/t - t}{1/t + t}$

$= \frac{\frac{1-t^2}{t}}{\frac{1+t^2}{t}}$ A

$= \frac{1-t^2}{1+t^2}$

$= \cos\theta$

5). $\frac{dr}{dt} = 0.03$

$\frac{dV}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{\cancel{3}}\pi \times \cancel{3}r^2 = 4\pi r^2$ B

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.03$

At $r = 3.2$ mm $\frac{dV}{dt} = 1.2288\pi$

6) Cases :- k-1st, Angela any of the 6 spots, Then place rest
$1 \times 6C_1 \times 5!$

k-2nd, Angela any of the 5 spots, Then place the rest.
$1 \times 5C_1 \times 5!$

following this pattern
Total $= 1\times 6C_1 \times 5! + 1\times 5C_1 \times 5! + 1 \times 4C_1 \times 5! + 1 \times 3C_1 \times 5! + 1 \times 2C_1 \times 5! + 1 \times 1C_1 \times 5!$

$= 5!(6C_1 + 5C_1 + 4C_1 + 3C_1 + 2C_1 + 1C_1)$
$= 2520$ C

7) $\underset{\sim}{s} + \underset{\sim}{t} = 4\underset{\sim}{i} + 9\underset{\sim}{j} + 5\underset{\sim}{i} + 3\underset{\sim}{j}$

$= 9\underset{\sim}{i} + 12\underset{\sim}{j}$ B

$|\underset{\sim}{s} + \underset{\sim}{t}| = \sqrt{9^2+12^2} = 15$

8) $\int \frac{a}{b+x^2} = 2\tan^{-1}\frac{x}{\sqrt{2}} + C$

$\int \frac{a}{b+x^2} = a\int \frac{1}{(\sqrt{b})^2+x^2}$

$= \frac{a}{\sqrt{b}}\tan^{-1} x/\sqrt{b} + C$

$\therefore \frac{a}{\sqrt{b}} = 2$
$\sqrt{b} = \sqrt{2}$ D
$b = 2$

$\frac{a}{\sqrt{2}} = 2$
$a = 2\sqrt{2}$

9) C

10) $(0, 0.75)$ C

11a) $P(x) = k(x-2)^3(x+3)$

Sub $(0,-4)$

$$-4 = k(0-2)^3(0+3)$$
$$-4 = k(-8)(3)$$
$$-4 = k \times -24$$
$$k = 4/24 = 1/6$$

$\therefore P(x) = \frac{1}{6}(x-2)^2(x+3$

11b). $(1-x)^5 = \binom{5}{0}(1)^5(-x)^0 + \binom{5}{1}(1)^4(-x)^1 + \binom{5}{2}(1)^3(-x)^2 + \binom{5}{3}(1)(-x)^3$
$+ \binom{5}{4}(1)^1(-x)^4 + \binom{5}{5}(1)^0(-x)^5$

$$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

$(1-x)^5(2x-3)^2 = (1-x)^5(4x^2-12x+9)$

Term with x^2 $= 1 \times 4x^2 + (5x)(12x) + (10x^2)9$
$= 4x^2 + 60x^2 + 90x^2$
$= 154x^2$

11c). $\sin 5x + \sin x = \sin 3x$.

$A + B = 5x$
$A - B = x$
$2A = 6x$
$A = 3x$
$B = 2x$

$2\sin 3x \cos 2x = \sin 3x$
$2\sin 3x \cos 2x - \sin 3x = 0$

$\sin 3x\,(2\cos 2x - 1) = 0$

$\sin 3x = 0$

$3x = 0, \pi, 2\pi, 3\pi$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

$\cos 2x = 1/2$

Acute $2x = \pi/6$

$2x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

d).

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

when $x = e$

$u = \ln e = 1$

$x = e^2$

$u = \ln e^2 = 2\ln e = 2$

$$\int_e^{e^2} \frac{2}{x(\ln x)^2}\,dx = 2\int_1^2 \frac{1}{u^2}\,du$$

$$= 2\left[-\frac{1}{u}\right]_1^2$$

$$= -2\left[\left(\frac{1}{2} - 1\right)\right]$$

$$= -1 + 2$$

$$= 1$$

e) (i) $E(X) = np = 100 \times 0.64 = 64$

$\sigma(X) = \sqrt{npq}$

$= \sqrt{100 \times 0.64 \times 0.36}$

$= 4.8$

(ii) z - score for $0.64 = 0$

z- score for $0.58 = \frac{58 - 64}{4.8} = -1.25$

$P(58 \leq X \leq 64)$

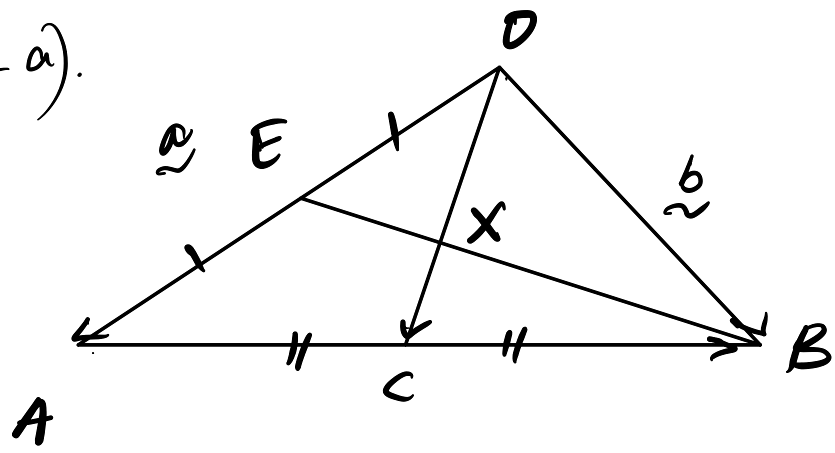
$= P(-1.25 \leq z \leq 0)$

$= P(z \leq 1.25) - 0.5$

$= 0.8944 - 0.5$

$= 0.3944$

12 a).



(i) $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AC} = 1/2\ \overrightarrow{AB}$

$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

(ii) $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$ (since $\overrightarrow{OC} \perp \overrightarrow{AB}$

$\frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$

$\frac{1}{2}[(\underset{\sim}{b} + \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})] = 0$

$\frac{1}{2}[\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a}] = 0$

$\frac{1}{2}[|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2] = 0$

$$|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2 = 0$$

$$\therefore \quad |\underset{\sim}{b}| = |\underset{\sim}{a}|$$

$$|\vec{OB}| = |\vec{OA}|$$

OB and OA are equal

b). $A[\sin(x+\alpha)] = 9\sin x + 40\cos x$

$A[\sin x\cos\alpha + \cos x\sin\alpha) = 9\sin x + 40\cos x$

$A\cos\alpha = 9$ —— ①

$A\sin\alpha = 40$ —— ②

② ÷ ① $\tan\alpha = 40/9$

$\alpha = \tan^{-1} 40/9$

$= 1.35$

$①^2 + ②^2 \quad A^2 = 9^2 + 40^2$

$A = \sqrt{9^2 + 40^2} = 41$

$\therefore \quad 9\sin\alpha + 4\cos\alpha = 41(\sin(x + 1.35))$

$41\sin(x + 1.35) = 6$

$\sin(x + 1.35) = 6/41$

Acute $(x + 1.35) = \sin^{-1} 6/41 = 0.147$

$0 \leq x \leq \pi$

$1.35 \leq x + 1.35 \leq 4.4916$

$\therefore \quad x + 1.35 = \pi - 0.147$

$= 2.99$

$x = 1.64$

c). $\frac{dy}{dx} = \frac{x^3+1}{xy}$

$$y\,dy = \frac{(x^3+1)}{x}\,dx$$

$$\int y\,dy = \int \frac{x^3}{x} + \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + \ln x + C$$

sub $(1, \frac{\sqrt{6}}{3})$

$$\left(\frac{\sqrt{6}}{3}\right)^2 / 2 = \frac{1^3}{3} + \ln(1) + C$$

$$\frac{6}{18} = \frac{1}{3} + C$$

$$\frac{1}{3} - \frac{1}{3} = C$$

$$C = 0$$

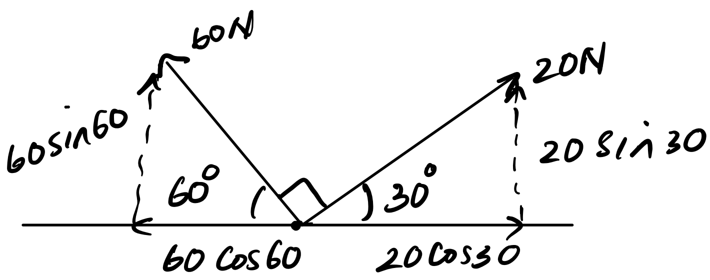
$$\frac{y^2}{2} = \frac{x^3}{3} + \frac{3\ln x}{3}$$

$$y^2 = \frac{2x^3 + 6\ln x}{3}$$

$$y = \sqrt{\frac{2x^3 + 6\ln x}{3}}$$

Since $y > 0$, when checking initial conditions.

d)



(i) $20\cos 30 - 60\cos 60$

$20 \times \frac{\sqrt{3}}{2} - 60 \times \frac{1}{2}$

$= 10\sqrt{3} - 30$

(ii) Vertically,

$20\sin 30 + 60\sin 60$

$= 20 \times \frac{1}{2} + 60 \times \frac{\sqrt{3}}{2}$

$= 10 + 30\sqrt{3}$

$\therefore$ Resultant $= (10\sqrt{3} - 30)\underset{\sim}{i} + (10 + 30\sqrt{3})\underset{\sim}{j}$

e). $y = 4x(x+5)^6$

$x = 4y(y+5)^6$

$\frac{dx}{dy} = 4y \cdot 6(y+5)^5 + (y+5)^6 \cdot 4$

$= 4(y+5)^5[6y + y + 5]$

$= 4(y+5)^5(7y+5)$

$\frac{dy}{dx} = \frac{1}{4(y+5)^5(7y+5)}$

13 a) $$f'(x) = \frac{1}{1+(e^{2x-1})^2} \times e^{2x-1} \times 2$$

$$= \frac{2e^{2x-1}}{1+(e^{2x-1})^2}$$

$$f'(1/2) = \frac{2e^{2\times 1/2 - 1}}{1+(e^{2\times 1/2 - 1})^2}$$

$$= \frac{2e^0}{1+(e^0)^2}$$

$$= \frac{2}{2} = 1$$

b) i)

$$8\cos^3\theta - 6\cos\theta - 1 = 0$$
$$8\cos^3\theta - 6\cos\theta = 1$$
$$4\cos^3\theta - 3\cos\theta = 1/2$$

comparing with $4\cos^3\theta - 3\cos\theta = \cos 3\theta$

$$\cos 3\theta = 1/2$$

(ii) $\cos 3\theta = 1/2$

Acute $3\theta = \pi/3$

$$3\theta = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3},\ 2\pi + \frac{\pi}{3},\ 2\pi + 2\pi - \frac{\pi}{3}$$

$$= \frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{11\pi}{3}$$

$$\theta = \frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \ldots$$

(first 3 distinct solution)

Roots of the equation are

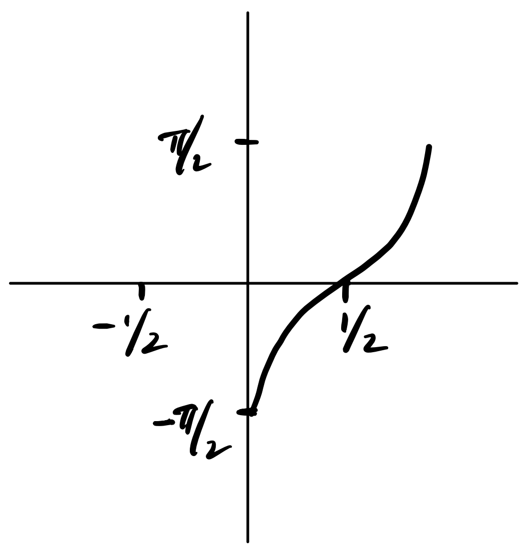
$x = \cos \pi/9,\ \cos 5\pi/9,\ \cos 7\pi/9.$

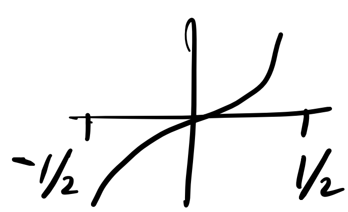
using product of roots

$$\cos \pi/9 \ \cos 5\pi/9 \ \cos 7\pi/9 = -d/a = 1/8$$

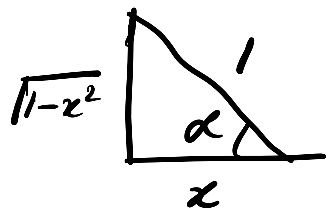
13 c)

$y = \sin^{-1} 2(x - 1/2)$





$$\sin^{-1}(2x-1) = \cos^{-1}x$$
$$2x-1 = \sin(\cos^{-1}x)$$
$$2x-1 = \sqrt{1-x^2}$$
$$(2x-1)^2 = 1-x^2$$
$$4x^2-4x+1 = 1-x^2$$
$$5x^2-4x = 0$$
$$x(5x-4) = 0$$
$$x = 0,\ 4/5$$



Since $x=0$ is not a solution, $x = 4/5$

d)

1) $\frac{5}{8}\left(\frac{1}{P} + \frac{1}{10000-P}\right)$

$$= \frac{5}{8}\left(\frac{10000-P+P}{P(10000-P)}\right)$$

$$= \frac{5}{8}\left(\frac{10000}{P(10000-P)}\right)$$

$$= \frac{50000}{8P(10000-P)}$$

$$= \frac{1}{\frac{8P}{50000}(10000-P)}$$

$$= \frac{1}{0.00016(10000-P)}$$

(i) $\frac{dP}{dt} = 0.00016P(10000 - P)$

$$\int \frac{dP}{0.00016P(10000-P)} = \int dt$$

$$\int \frac{5}{8}\left(\frac{1}{P} + \frac{1}{10000-P}\right) = \int dt$$

$$\frac{5}{8}\left[\ln|P| - \ln|10000-P|\right] = t + c$$

$$\frac{5}{8}\left[\ln\left|\frac{P}{10000-P}\right|\right] = t + c$$

At $t=0$, $P = 2500$

$$\frac{5}{8}\left[\ln \frac{2500}{7500}\right] = 0 + c$$

$$\frac{5}{8}\left[\ln \frac{1}{3}\right] = c$$

$$\therefore \frac{5}{8}\left[\ln\left|\frac{P}{10000-P}\right|\right] = t + \frac{5}{8}\ln\frac{1}{3}$$

$$\ln\left|\frac{P}{1000-P}\right| = \frac{8}{5}t + \cancel{\frac{5}{8}} \times \cancel{\frac{8}{5}} \ln 1/3$$

$$\ln\left|\frac{P}{1000-P}\right| = 1.6t + \ln 1/3$$

$$\left|\frac{P}{10000-P}\right| = e^{1.6t + \ln 1/3}$$

$$\left|\frac{P}{10000-P}\right| = e^{1.6t} \times e^{\ln 1/3}$$

$$\left|\frac{P}{10000-P}\right| = e^{1.6t} \times \frac{1}{3}$$

$$\pm \frac{P}{10000-P} = e^{1.6t} \times \frac{1}{3}$$

Checking initial condition

$$\frac{P}{10000-P} = e^{1.6t} \times \frac{1}{3}$$

$$\frac{3P}{10000-P} = e^{1.6t}$$

$$3P = e^{1.6t}(10000 - P)$$

$$3P = 10000\,e^{1.6t} - Pe^{1.6t}$$

$$3P + Pe^{1.6t} = 10000\,e^{1.6t}$$

$$P(3 + e^{1.6t}) = 10000\,e^{1.6t}$$

$$P = \frac{10000\,e^{1.6t}}{3 + e^{1.6t}}$$

Dividing through out $e^{1.6t}$

$$P = \frac{10000}{3e^{-1.6t} + 1}$$

Q14) a). Prove the statement is true for $n=1$

LHS $= 1$

RHS $= \frac{1}{4}(5^1 - 1) = 1$

Assume true for $n=k$

$1+5+25+ \cdots\cdots +5^{k-1} = \frac{1}{4}(5^k - 1)$

Prove true for $n=k+1$ if true for $n=k$

RTP: $1+5+25+ \cdots\cdots +5^k = \frac{1}{4}(5^{k+1} - 1)$

<u>LHS</u>

$1+5+25+ \cdots\cdots 5^{k-1} + 5^k$

$= \frac{1}{4}(5^k - 1) + 5^k$ (using $S(k)$)

$= \frac{1}{4}5^k - \frac{1}{4} + \frac{5^k \times 4}{4}$

$= \frac{1}{4}(5^k - 1 + 4 \times 5^k)$

$= \frac{1}{4}(5^1 \times 5^k - 1)$

$= \frac{1}{4}(5^{k+1} - 1)$

$=$ RHS

$\therefore$ Since the statement is true for $n=1$, & it's true for $n=k+1$ if true for $n=k$, by the principle of mathematical induction it's true for all integers $n \geq 1$

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b). $\frac{dx}{dt} = 10$

$$k^2 = 0.7^2 + x^2$$
$$k = \sqrt{0.7^2 + x^2}$$
$$\frac{dk}{dx} = \frac{1}{\cancel{2}\sqrt{0.7^2+x^2}} \times \cancel{2}x$$

$$\frac{dk}{dt} = \frac{dk}{dx} \times \frac{dx}{dt}$$
$$= \frac{x}{\sqrt{0.7^2+x^2}} \times 10 = \frac{10x}{\sqrt{0.7^2+x^2}}$$

(ii) $\tan\theta = \frac{10}{k}$

$$k = \frac{10}{\tan\theta}$$

$$k = 10(\tan\theta)^{-1}$$

$$\frac{dk}{d\theta} = -10(\tan\theta)^{-2} \times \sec^2\theta$$

$$= \frac{-10\sec^2\theta}{\tan^2\theta}$$

$$= \frac{-10 \times \times \cancel{\cos^2\theta}}{\cancel{\cos^2\theta} \times \sin^2\theta} = -10\operatorname{cosec}^2\theta.$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dk} \times \frac{dk}{dt}$$

$$= \frac{1}{-10\operatorname{cosec}^2\theta} \times \frac{10x}{\sqrt{0.7^2+x^2}}$$

When $x = 2.4$

$$k = \sqrt{0.7^2+2.4^2} = 5/2$$

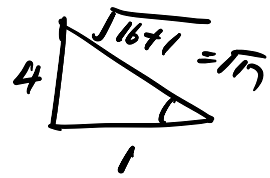
$$\theta = \tan^{-1}\frac{10}{k}$$

$$= \tan^{-1}\frac{10}{5/2}$$

$$= \tan^{-1} 4$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{-10\operatorname{cosec}^2\left(\tan^{-1}(4)\right)} \times \frac{10(2.4)}{\sqrt{0.7^2+2.4^2}}$$

$$= \frac{1}{-10\times\left(\frac{\sqrt{17}}{4}\right)^2} \times \frac{24}{\frac{5}{2}}$$



$$= \frac{-1}{\frac{85}{8}} \times \frac{24\times 2}{5}$$

$$= \frac{-8\times 24\times 2}{85\times 5}$$

$$= \frac{-384}{425} \text{ rad/s}$$

c)(i) Equating x-values.

$$5000 = 400t\cos\theta$$

$$t\cos\theta = \frac{5000}{400} = 12.5$$

Equating y-values.

$$3600 + 400t\sin\theta - \cancel{5t^2} = 5000 - \cancel{5t^2}$$

$$400t\sin\theta = 1400$$

$$t\sin\theta = \frac{1400}{400} = 7/2$$

$$\frac{t\sin\theta}{t\cos\theta} = \frac{7/2}{12.5}$$

$$\tan\theta = 0.28$$

$$\theta = \tan^{-1}(0.28) = 15^\circ 39'$$

(ii) $t\cos\theta = 12.5$

$$t\cos(15^\circ 39) = 12.5$$

$$t = 12.5/\cos 15^\circ 39'$$

$$= 12.98\ s$$

$$y = 5000 - 5t^2$$

$$= 5000 - 5(12.98)^2$$

$$= 4157.6$$

Differentiating y-component of displacement

$$v = -10t$$

$$= -10(12.98)$$

$$= -129.8$$

$$= -130$$

Speed $= 130$ m/s